Collatz's conjecture on congruous numbers 3 mod 4 through custom sequence

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Abstract

In this article, which follows my previous works, the dynamics of the custom sequence associated with the Collatz conjecture for odd numbers N congruous 3 mod.4 is explored, in which in each **iteration (**understood as a set of odd steps and even steps dependent on the initial number of the same iteration) the next odd number is reached which becomes the initial number of the next iteration. This structure allows each iteration to be classified into four distinct cases (3-4-5-6), based on the decomposition of the initial number. Cases 3 and 4 are always descending (when the initial number N of the iteration is congruous 1 mod. 4), while cases 5 and 6 can be either descending or ascending (the initial number N of the iteration is congruous 3 mod. 4). It is shown that all the initial numbers that generate ascending iterations belong to only two canonical families (C and D) of congruous numbers 3 mod.4 and that these families cannot produce ascending iterations indefinitely. It is shown that the percentage of numbers belonging to the non-descending C and D families in the first iterations is reduced to a few units and it is shown that each number belonging to the C and D families must necessarily enter a descending case (**geometric decay**) so that in subsequent iterations a value lower than the initial one is obtained.

1. Introduction

The canonical Collatz sequence is defined for every positive integer as follows:N

- If it is even: $N \to N/2$
- If it is odd: $N \to 3N + 1$

The conjecture states that, regardless of the initial value N, the sequence always converges to **1**. Despite its apparent simplicity, no general proof has yet been found.
In this article we explore the custom conjecture equivalent to the canonical Collatz conjecture for odd numbers N congruous 3 mod. 4 of cases 5 and 6 in which each number N can be written in the form: N = 2^p*(1 + 2k) – 1 and depending on the parity of p and k, the compound iteration belongs to case 5 with p and k having the same parity e or to case 6 with p e k having opposite parity .

2. Classification of cases 5 and 6 into 4 families of odd numbers congruous 3 mod.4

To study ascending iterations, a classification into four families of odd numbers congruous 3 mod.4 is introduced, based on the 2-adic structure and on the congruence modulo 4.
Family **A**: N = 2^p*(1 + 4w) – 1, with p even, p ≥ 2, k even and k=2*w with w>=0;
Family **B**: N = 2^p*(3 + 4w) – 1, with odd p, p ≥ 3, odd k and k=1+2*w with w>=0;
Family **C**: N = 2^p*(1 + 4w) – 1, with odd p, p ≥ 3, k even and k=2*w with w>=0;
Family **D**: N = 2^p*(3 + 4w) – 1, with even p, p ≥ 2, odd k and k=1+2*w with w>=0.
So families **A** and **B** relate to case 5, and families **C** and **D** relate to case 6 of the personalized succession.
The following tables list the numbers N odd congruous 3 mod.4 between 1 and 2^20=1048576 divided by cases 3 – 4 – 5 – 6 and by families A – B – C – D.

Case 3 with odd k with k=1+2*w and with N[0]=4*(1+2*w)+1=8*w+5

N[0]=4*(1+2*w)+1	How many out of 262,144 odd numbers are congruent to 1 mod 4	N[1]=2*w+1	N[1] decreases or increases compared to N[0]	Probability on 1,048,576 numbers considered	odd step	even step
8*w+5	0<=w<=131071	2*w+1	Decreases	12.5000000000%	0	2
Total	131072		Total	12.5000000000%		

Case 4 with even k with k=2*w and with N[0]=4*(2*w)+1=8*w+1

N[0]=4*(2*w)+1	How many out of 262,144 odd numbers are congruent to 1 mod 4	N[1]=6*w+1	N[1] decreases or increases compared to N[0]	Probability on 1,048,576 numbers considered	odd step	even step
8*w+1	0<=w<=131071	6*w+1	Decreases	12.5000000000%	1	2
Total	131072		Total	12.5000000000%		

Family A - Case 5 with even p and even k with k=2*w so (1+2*k)=(1+4*w) with N[0]=2^p*(1+4*w)-1

p=	N[0]=2^p*(1+4*w)-1	How many out of 262,144 odd numbers are congruent to 1 mod 4	N[1]=(3^(p-1)*(1+4*w)-1)/2	N[1] decreases or increases compared to N[0]	Probability on 1,048,576 numbers considered	odd step	even step
2	16*w+3	0<=w<=65535	6*w+1	Decreases	6.2500000000%	1	3
4	64*w+15	0<=w<=16383	54*w+13	Decreases	1.5625000000%	3	5
6	256*w+63	0<=w<=4095	486*w+121	Increase	0.3906250000%	5	7
8	1024*w+255	0<=w<=1023	4374*w+1093	Increase	0.0976562500%	7	9
10	4096*w+1023	0<=w<=255	39366*w+9841	Increase	0.0244140625%	9	11
12	16384*w+4095	0<=w<=63	354294*w+88573	Increase	0.0061035156%	11	13
14	65536*w+16383	0<=w<=15	3188646*w+797161	Increase	0.0015258789%	13	15
16	262144*w+65535	0<=w<=3	28697814*w+7174453	Increase	0.0003814697%	15	17
18	1048576*w+262143	w=0	258280326*w+64570081	Increase	0.0000953674%	17	19
20	4194304*w+1048575	w=0	2324522934*w+581130733	Increase	0.0000953674%	19	21
	Total	87382		Total	8.3333969116%		

Family B - Case 5 with odd p and odd k = 1+2*w so (1+2*k) = (3+4*w) with N[0] = 2^p * (3+4*w) - 1

p=	N[0]=2^p*(3+4*w)-1	How many out of 262,144 odd numbers are congruent to 1 mod 4	N[1]=(3^(p-1)*(3+4*w)-1)/2	N[1] decreases or increases compared to N[0]	Probability on 1,048,576 numbers considered	odd step	even step
3	32*w+23	0<=w<=32767	18*w+13	Decreases	3.1250000000%	2	4
5	128*w+95	0<=w<=8191	162*w+121	Increase	0.7812500000%	4	6
7	512*w+383	0<=w<=2047	1458*w+1093	Increase	0.1953125000%	6	8
9	2048*w+1535	0<=w<=511	13122*w+9841	Increase	0.0488281250%	8	10
11	8192*w+6143	0<=w<=127	118098*w+88573	Increase	0.0122070313%	10	12
13	32768*w+24575	0<=w<=31	1062882*w+797161	Increase	0.0030517578%	12	14
15	131072*w+98303	0<=w<=7	9565938*w+7174453	Increase	0.0007629395%	14	16
17	524288*w+393215	0<=w<=1	86093442*w+64570081	Increase	0.0001907349%	16	18
	Total	43690		Total	4.1666030884%		

Family C - Case 6 with odd p and even k with k=2*w so (1+2*k)=(1+4*w) with N[0]=2^p*(1+4*w)-1

p=	N[0]=2^p*(1+4*w)-1	How many out of 262,144 odd numbers are congruent to 1 mod 4	N[1]=(3^(p)*(1+4*w)-1)/2	N[1] decreases or increases compared to N[0]	Probability on 1,048,576 numbers considered	odd step	even step
3	32*w+7	0<=w<=32767	54*w+13	Increase	3.1250000000%	3	4
5	128*w+31	0<=w<=8191	486*w+121	Increase	0.7812500000%	5	6
7	512*w+127	0<=w<=2047	4374*w+1093	Increase	0.1953125000%	7	8
9	2048*w+511	0<=w<=511	39366*w+9841	Increase	0.0488281250%	9	10
11	8192*w+2047	0<=w<=127	354294*w+88573	Increase	0.0122070313%	11	12
13	32768*w+8191	0<=w<=31	3188646*w+797161	Increase	0.0030517578%	13	14
15	131072*w+32767	0<=w<=7	28697814*w+7174453	Increase	0.0007629395%	15	16
17	524288*w+131071	0<=w<=1	258280326*w+64570081	Increase	0.0001907349%	17	18
19	2097152*w+524287	w=0	2324522934*w+581130733	Increase	0.0000953674%	19	20
	Total	43691		Total	4.1666984558%		

Family D - Case 6 with even p and odd k with k=1+2*w so (1+2*k)=(3+4*w) with N[0]=2^p*(3+4*w)-1

p=	N[0]=2^p*(3+4*w)-1	How many out of 262,144 odd numbers are congruent to 1 mod 4	N[1]=(3^(p)*(3+4*w)-1)/2	N[1] decreases or increases compared to N[0]	Probability on 1,048,576 numbers considered	odd step	even step
2	16*w+11	0<=w<=65535	18*w+13	Increase	6.2500000000%	2	3
4	64*w+47	0<=w<=16383	162*w+121	Increase	1.5625000000%	4	5
6	256*w+191	0<=w<=4095	1458*w+1093	Increase	0.3906250000%	6	7
8	1024*w+767	0<=w<=1023	13122*w+9841	Increase	0.0976562500%	8	9
10	4096*w+3071	0<=w<=255	118098*w+88573	Increase	0.0244140625%	10	11
12	16384*w+12287	0<=w<=63	1062882*w+797161	Increase	0.0061035156%	12	13
14	65536*w+49151	0<=w<=15	9565938*w+7174453	Increase	0.0015258789%	14	15
16	262144*w+196607	0<=w<=3	86093442*w+64570081	Increase	0.0003814697%	16	17
18	1048576*w+786431	w=0	774840978*w+581130733	Increase	0.0000953674%	18	19
	Total	87381		Total	8.3333015442%		
Total numbers N congruent to 1 and 3 mod 4-->	524288		Total numbers N congruent to 1 and 3 mod 4-->	50.00%			

In these tables you can see:

1. how the initial number N[0] and its subsequent N[1] depend on the value of w>=0;
2. the quantity of N[0] and N[1] for each value of p and the respective probability on the total of 1048576 odd and even numbers;
3. if N[1] increases or decreases with respect to N[0] based on the dependence of N[0] and N[1] on w;
4. in the family A with p=2 and with p=4 the value of N[1] decreases with respect to the value of N[0];
5. in the family B with p=3 the value of N[1] decreases with respect to the value of N[0];
6. in all other cases of families A – B – C – D the value of N[1] increases with respect to the value of N[0].

3. Reduction of families A – B – C – D to families C – D.

Remark 3.1
If we compare the values of N[1] in the tables above, we see that they are the same even if they derive from different values of N[0]. In fact, in the family A with p=2 the value of N[1]=6*w+1 coincides with the value N[1] which derives from N[0]=8*w+1 of case 4. Furthermore, any other value of N[1] of family A with even p and p>=4 coincides with the value of N[1] that derives from N[0] belonging to family C with odd p and p>=3. And finally we also have that every other value of N[1] of the family B with odd p and p>=3 coincides with the value of N[1] which derives from N[0] belonging to the family D with even p and p>=2.
In fact, this is the case with theorem 3.1 below.

Theorem 3.1
Any number N[0] congruous 3 mod.4 with N[0]=2^p*(1+2*k)-1 for every p>=2 and for every k>=0 is isopath to the number M[0] congruous 3 mod.4 with M[0]=2^(p-1) *(1+2*k)-1 where M[0]<N[0].

Proof. Let N[0]=2^p*(1+2*k)-1 with p and k having the same parity (family A or B) and let M[0]=2^(p-1)*(1+2*k)-1 with p-1 and k having different parity (family C or D). For the custom sequence we have that N[0] is a case 5 and therefore N[1]=(3^(p-1)*(1+2*k)-1)/2, while M[0] is a case 6 for which M[1]=(3^(p-1)*(1+2*k)-1)/2 and therefore N[1]= M[1] for which N[0] and M[0] are isopaths because in their first iteration of the custom sequence they converge to the same number N[1]=M[1]=(3^(p-1)*(1+2*k)-1)/2 with p>=2 for every odd or even p and for every integer k>=0 and since 2^(p-1)*(1+2*k)-1<2^p*(1+2*k)-1 we have that M[0]<N[0]. **C.v.d.**

Remark 3.2
Therefore, considering observation 3.1 and theorem 3.1, we have that every number of the family A with p>=2 and w>=0 has a corresponding minor number and isopath belonging to it to the family C with p-1>=1 and the same w>=0 and every number of the family B with p>=3 and w>=0 has a corresponding minor number and isopath belonging to it to the family D with p-1>=2 and w>=0 and therefore if every number of the family C or D satisfies the Collatz conjecture then every number of the family A or B also satisfies the Collatz conjecture.

Example 1. Let N[0]=2^p*(1+2*k)-1 be a number of the family A case 5 with p=4 and k even and k=2*w with w>=0 then N[0]=2^4*(1+2*2*w)-1=64*w+15 to which, after the 1st iteration of the custom sequence, corresponds N[1]=(3^(p-1) *(1+2*k)-1)/2=(3^3*(1+2*2*w)-1)/2=54*w+13 with w>=0. Let M[0] be a number of the family C case 6 with p-1=3 and the same value as k even and k=2*w with w>=0 for which M[0]= 2^3*(1+2*2*w)-1=32*w+7 to which, after the 1st iteration of the custom sequence, corresponds M[1]=(3^(p-1) *(1+2*k)-1)/2=(3^3*(1+2*2*w)-1)/2=54*w+13 with w>=0. Therefore N[1]=M[1] and M[0]<N[0].
Example 2. Let N[0]=2^p*(1+2*k)-1 be a number of the family B case 5 with p=3 and odd k and k=1+2*w with w>=0 then N[0]=2^3*(3+2*2*w)-1=32*w+23 to which, after the 1st iteration of the custom sequence, corresponds N[1]=(3^(p-1) *(1+2*k)-1)/2=(3^2*(3+2*2*w)-1)/2=18*w+13 with w>=0. Let M[0]

be a number of the family D case 6 with p-1=2 and the same value as odd k and k=1+2*w with w>=0 for which M[0]= 2^2*(3+2*2*w)-1=16*w+11 to which, after the 1st iteration of the custom sequence, corresponds M[1]=(3^(p-1) *(1+2*k)-1)/2=(3^2*(3+2*2*w)-1)/2=18*w+13 with w>=0. So N[1]=M[1] and M[0]<N[0].

4. Further reduction in the numbers belonging to families C – D.

From the previous paragraphs and tables we can now calculate the percentages of the numbers to which they correspond in the custom sequence of isopath numbers and less than them.
Let's consider a group of positive and consecutive integers, both odd and even, between **1** and **2^n** with **very** large and even n and therefore there are 2^(n-1) even numbers, there are 2^(n-2) odd numbers congruous 1 mod.4 (case 3 and 4) and there are 2^(n-2) odd numbers congruous 3 mod.4 (case 5 and 6 i.e. the numbers belonging to the 4 families A-B-C-D). So 2^(n-3) are the numbers belonging to the families A+B and 2^(n-3) are the numbers belonging to the families C+D. Furthermore, since n is even in the family D, we have that 2<=p<=n-2 and therefore the total of the numbers belonging to the family D are 2^(n-4) for p=2 (i.e. for p=2 we already have half the numbers of the families C+D), they are 2^(n-6) for p=4 and decrease by 1/4 for every increase of 2 in the value of p up to p=n-2 in which there is only one number N corresponding to the value w=0. Therefore the total of the numbers belonging to the family D are 2/3*(2^(n-3)+1)-1 and that is 2/3 of the numbers belonging to the families C and D together. On the other hand, in the C family, being n even, we have 3<=p<=n-1 so the numbers belonging to the C family are 2^(n-5) for p=3, they are 2^(n-7) for p=5 and decrease by 1/4 for each increase of 2 in the value of p up to p=n-1 in which there is only one number N corresponding to the value w=0. Therefore the total of the numbers belonging to the C family is 1/3*(2^(n-3)+1) and that is 1/3 of the numbers belonging to the C and D families together. In fact, for n=20 we have from the tables considered in paragraph 2 that the total number of numbers belonging to family C is 1/3*(2^(n-3)+1)= 1/3*(2^(20-3)+1)= 43691 while the total number of numbers belonging to family D is 2/3*(2^(n-3)+1)-1=2/3*(2^(20-3)+1)-1=87381 so that the total number of numbers belonging to family C and D are 43691+87381= 131072 = 2^(20-3) for n=20.
We know that all even numbers and all odd numbers congruous 1 mod.4 after the first iteration have a minor number and isopath to them and all numbers of families A and B have a minor number and isopath to them in a corresponding number belonging to families C and D respectively. they are a total of 2^(n-3) out of the total of 2^n numbers considered, i.e. **12.5% of all the numbers considered (in fact 2^(n-3)/2^n =2^(-3)=1/8=0.125)**. **But all the numbers belonging to the 12.5% of the C and D families have very different probabilities of occurring in the custom sequence (the probability determines the total of them in the respective family based on the value of p and this decreases by 1/4 each time the value of p increases by 2 in each of the C and D families) and these probability values (for each value of p) are the same whatever the group of numbers 2^n considered, i.e. these probability values attributed to each value of p in the families C and D do not depend on the chosen value of n.**
In fact, as can be seen from the tables considered above, for **n=20** the probabilities with which the numbers occur decrease with respect to the increase in p, i.e. the probabilities of the numbers decrease by 1/4 with respect to the probability of the previous numbers in each family. In fact, in the **family D** for **p=2 there is a probability of 6.25%** (1/2 of the numbers belonging to the total of the families C and D together) and for **p=4 there is a probability of 1.5625%** (1/4 of the previous one) and in the family **C** for **p=3 there is a probability of 3.125%** (1/4 of the numbers belonging to the total of families C and D together) and for **p=5 there is a probability of 0.78125%** (1/4 of the previous one) C and D. moreover, if we consider only the values of **2<=p<=5** (between 2 and 5) we arrive at the probability of **11.71875%** for the numbers of families C and D**.** For **p=19** of family **C there is a probability of 0.0000953674%** and for **p=18** of family **D there is a probability of 0.0000953674%.**
With the following theorems we can decrease the **12.5%** of the numbers of the C and D families that increase the value of N[0] in the first iterations of the custom sequence.

Note 4.1 In the following, according to each following theorem on the C and D families, we will always eliminate from **12.5%** that percentage of initial N[0] numbers that satisfy one of the following options:

1. is isopath to a number M[0] with M[0]<N[0];
2. after two iterations of the custom sequence it converges to N[2]<N[0], i.e. ***it slides*** during the sequence.

Theorem 4.1 Every number N[0] congruous 3 mod.4 with N[0]=2^2*(1+2*k)-1 of the family D (case 6 with p=2 and with odd k and k=1+2*w and with even w and w=2*t for every t>=0) after 2 iterations of the custom sequence turns into N[2]=9*t+3 with 2 odd steps and 5 even steps if t is even or in N[2]=27*t+10 with 3 odd passes and 5 even passes if t is odd where N[2]<N[0].
Proof. Let N[0]=2^2*(1+2*k)-1 of the family D (case 6 with p=2 and with odd k and k=1+2*w and with even w and w=2*t for every w>=0) for which N[0]=2^2*(1+2*k)-1=2^2*(1+2*(1+4*t))-1=4*(3+8*t))-1=32*t+11 which being a case 6 is transformed after 1 iteration in N[1]=(3^2*(3+8*t)-1)/2= 36*t+13 i.e. N[1]=4*(9*t+3)+1 where it is a case 3 if t is even (for which 9*t+3 is odd) or a case 4 if t is odd (for which 9*t+3 is even) and therefore in the 2nd iteration of the custom sequence we have that N[2]= 9*t+3 if t is even or N[2]= 3*(9*t+3)+1=27*t+10 if t is odd and in both cases comparing initial N[0] with N[2] we always have N[2]<N[0]. **C.v.d.**

Theorem 4.2 Any number N[0] congruous 3 mod.4 of the family D (case 6 with p>=2 and with odd k and k=1+2*w) with odd w and multiple of 3 i.e. w=3*(1+2*t) for every t>=0 for which w=3+6*t is odd is isopath to the number M[0]=2^(p+1)*(1+2*(2+4*t))-1 of the C family case 6 with odd p+1 and p+1>=3 and with k even and k=2*(1+2*t)=2+4*t and we have M[0]<N[0].
Proof. Let N[0]=2^p*(3+4*w)-1=2^p*(3+4*(3+6*t))-1=2^p*(15+24*t)-1=2^p*3*(5+8*t)-1 of the family D (case 6 with p>=2 and with odd k and k=1+2*w) and with odd w and multiple of 3 i.e. w=3*(1+2*t)=3+6*t for every t>=0 and since N[0] is In case 6, it turns into N[1]= (3^p*(3+4*(3+6*t))-1)/2= (3^(p+1)*(1+4*(1+2*t))-1)/2=(3^(p+1)*(8*t+5)-1)/2 for every t>=0. Consider the number M[0] of the family C case 6 with p+1>=3 odd and with k even and k=2*(1+2*t) for which M[0]=2^(p+1)*(1+2*k)-1=2^(p+1)*(1+2*2*(1+2*t))-1=2^(p+1)*(5+8*t))-1 and since M[0] is a case 6 it is transformed in the first iteration into M[1]=(3^(p+1)*(1+2*2*(1+2*t))-1)/2=(3^(p+1)*(8*t+5)-1)/2 i.e. N[1]=M[1] for which N[0] and M[0] are isopaths to each other because after the first iteration in their respective custom sequences both converge to N[1]=M[1] and also since 2^(p+1)*(5+8*t))-1<2^p*3*(5+8*t)-1 we have M[0] < N[0] for every value of t>=0. **C.v.d.**

Theorem 4.3 Any number N[0] congruous 3 mod.4 with N[0]=2^p*(3+4*w)-1 of the family D case 6 with even p and p>=4 and with odd k and k=1+2*w with odd or even w and multiple of 3 i.e. w=3*t for every t>=0 is isopath to the number M[0]=2^(p+1)*(1+2*(2*t))-1 of the family C case 6 with p+1 odd and p+1>=5 and with k even and k=2*t and we have M[0]<N[0].
Proof. Let N[0]= 2^p*(3+4*w)-1 of the family D case 6 with even p and p>=4 and with odd k and k=1+2*w with odd or even w and multiple of 3 i.e. w=3*t for every t>=0 so that after the necessary substitutions we have N[0]= 2^p*(3+4*3*t)-1]= 2^p*(3*(1+4*t))-1=3*2^p*(1+4*t))-1 and since N[0] is a case 6 it is transformed in the first iteration into N[1]=(3^p*(3*(1+4*t))-1)/2=(3^(p+1)*(1+4*t))-1)/2. Consider the number M[0] of the family C case 6 with p+1 odd and p+1>=5 and with k even and k=2*t with t even or odd and for every t>=0 for which M[0]=2^(p+1)*(1+4*t)-1 and since M[0] is a case 6 it is transformed in the first iteration into M[1]=(3^(p+1)*(1+4*t)-1)/2 for which results in M[1]=(N[1]. Therefore N[0] and M[0] are isopaths to each other because after the first iteration in their respective custom sequences both converge to N[1]=M[1]= =(3^(p+1)*(1+4*t)-1)/2 and since 2^(p+1)*(1+4*t)-1<3*2^p*(1+4*t))-1 we have M[0]<N[0]. **C.v.d.**

Theorem 4.4 Every number N[0] congruous 3 mod.4 with N[0]=2^4*(3+4*w)-1 of the family D case 6 with even p and p=4 and with odd k and k=1+2*w with even w and w=2+12*t for every t>=0 after 2 iterations of the custom sequence is transformed into N[2]=486*t+111 with 4 odd steps and 7 even steps and has N[2]<N[0].

Proof. Let N[0]=2^4*(3+4*w)-1 of the family D case 6 with even p and p=4 and with odd k and k=1+2*w with even w and w=2+12*t for every t>=0 for which N[0]=16*(3+4*(2+12*t))-1=768*t+175 and since N[0] is a case 6 it transforms after the 1 iteration with 4 odd steps and 5 even steps in N[1]=(3^4*(3+4*(2+12*t))-1)/2=1944*t+445 i.e. N[1]=4*(486*t+111)+1 where it is a case 3 with 486*t+111 odd and therefore in the 2nd iteration of the custom sequence with 0 odd steps and 2 even steps we have that N[2]=486*t+111 and since 486*t+111<768*t+175 we have that N[2]<N[0] for every t>=0 and also in the sequence from N[0] to N[2] there are with 4 odd steps and 7 even steps. **C.v.d.**

Theorem 4.5 Every number N[0] congruous 3 mod.4 with N[0]=2^4*(3+4*w)-1 of the family D case 6 with even p and p=4 and with odd k and k=1+2*w with even w and w=10+12*t for every t>=0 after 2 iterations of the custom sequence with 4 odd steps and 7 even steps turns into N[2]= 486*t+435 and has N[2]<N[0].
Proof. Let N[0]=2^4*(3+4*w)-1 of the family D case 6 with even p and p=4 and with odd k and k=1+2*w with even w and w=10+12*t for every t>=0 for which N[0]=16*(3+4*(10+12*t))-1=768*t+687 and since N[0] is a case 6 it transforms after the 1 iteration with 4 odd steps and 5 even steps in N[1]=(3^4*(3+4*(10+12*t))-1)/2=24*81*t+1741 i.e. N[1]=4*(6*81*t+435)+1 where it is a case 3 with odd 6*81*t+435 and therefore in the 2nd iteration of the custom sequence with 0 odd steps and 2 even steps we have that N[2]= 6*81*t+435 =486*t+435 and since 486*t+435 <768* t+687 we have that N[2]<N[0] for every t>=0 and also in the sequence from N[0] to N[2] there are with 4 odd passages and 7 even passages. **C.v.d.**

Theorem 4.6 Every number N[0] congruous 3 mod.4 with N[0]=2^3*(1+4*w)-1 of the family C case 6 with odd p and p=3 and with k even and k=2*w with even w and w=4*t for every t>=0 after 2 iterations of the custom sequence with 3 odd steps and 6 even steps turns into N[2]= 54*t+3 and we have N[2]<N[0].
Proof. Let N[0]=2^3*(1+4*w)-1 of the family C case 6 with odd p and p=3 and with even k and k=2*w with even w and w=4*t for every t>=0 for which N[0]=8*(1+4*(4*t))-1=128*t+7 and since N[0] is a case 6 it transforms after the 1st iteration with 3 odd steps and 4 even passes in N[1]=(3^3*(1+4*(4*t))-1)/2=8*27*t+13 i.e. N[1]=4*(2*27*t+3)+1 where it is a case 3 with 2*27*t+3 odd and therefore in the 2nd iteration of the custom sequence with 0 odd steps and 2 even steps we have that N[2]= 2*27*t+3 =54*t+3 and since 54*t+3<128*t+7 we have that N[2]<N[0] for every t>=0 and also in the sequence from N[0] to N[2] there are with 3 odd passages and 6 even passages. **C.v.d.**

Theorem 4.7 Every number N[0] congruous 3 mod.4 with N[0]=2^5*(1+4*w)-1 of the family C case 6 with odd p and p=5 and with k even and k=2*w with even w and w=2+4*t for every t>=0 after 2 iterations of the custom sequence with 5 odd steps and 8 even steps it turns into N[2]= 486*t+273 and we have N[2]<N[0].
Proof. Let N[0]=2^5*(1+4*w)-1 of the family C case 6 with odd p and p=5 and with even k and k=2*w with even w and w=2+4*t for every t>=0 for which N[0]=32*(1+4*(2+4*t))-1=512*t+287 and since N[0] is a case 6 it transforms after the 1 iteration with 5 odd steps and 6 even steps in N[1]=(3^5*(1+4*(2+4*t))-1)/2=8*243*t+1093 i.e. N[1]=4*(2*243*t+273)+1 where it is a case 3 with 2*243*t+273 odd and therefore in the 2nd iteration of the custom sequence with 0 odd steps and 2 even steps we have that N[2]= 2*243*t+273 =486*t+273 and since 486*t+273 <512*t+287 we have that N[2]<N[0] for every t>=0 and also in the sequence from N[0] to N[2] there are with 5 odd passages and 8 even passages. **C.v.d.**

Note 4.2 As anticipated in **Note 4.1** and as a consequence of **theorems 4.1 to 4.7** in the following tables for the group of numbers from 1 to 2^20 we have, on the basis of the previous theorems, the quantity and probability of the numbers satisfying the same theorem.

Theorem	Probability
Theorem 4.1	3.12500000000000000%
Theorem 4.2	1.3884544372558600%
Theorem 4.3	0.3475189208984370%
Theorem 4.4	0.1302719116210940%
Theorem 4.5	0.1301765441894530%
Theorem 4.6	0.78125000000000000%
Theorem 4.7	0.19531250000000000%
Total probability	6.097984313964840%

Family C - Case 6 with odd p and even k with k=2*w so (1+2*k)=(1+4*w) with N[0]=2^p*(1+4*w)-1					
Total numbers for family C	43691	Sliding numbers		Probability of Slipping Numbers	
p=	Probability on 1,048,576 numbers considered	Theorem 4.6 w=0+4*t	Theorem 4.7 w=2+4*t	Theorem 4.6 w=0+4*t	Theorem 4.7 w=2+4*t
3	3.1250000000%	8192		0.781250%	
5	0.7812500000%		2048		0.19531250%
7	0.1953125000%				
9	0.0488281250%				
11	0.0122070313%				
13	0.0030517578%				
15	0.0007629395%				
17	0.0001907349%				
19	0.0000953674%				
Total	4.1666984558%	8192	2048	0.78125000%	0.19531250%
		Total numbers sliding --->		10240	
		Total probability family C-->		0.97656250%	

Family D - Case 6 with even p and odd k with k=1+2*w so (1+2*k)=(3+4*w) with N=2^p*(3+4*w)-1											
Total numbers for family D	87381	Sliding numbers					Probability of Slipping Numbers				
p=	Probability over 1,048,576 numbers considered	Theorem 4.2 w=3+6*t	Theorem 4.3 w=0+3*t	Theorem 4.4 w=2+12*t	Theorem 4.5 w=10+12*t	Theorem 4.1 w=0+2*t	Theorem 4.2 w=3+6*t	Theorem 4.3 w=0+3*t	Theorem 4.4 w=2+12*t	Theorem 4.5 w=10+12*t	Theorem 4.1 w=0+2*t
2	6.25000000000000%	10922				32768	1.04160308837890%				3.12500000000000%
4	1.56250000000000%	2730	2731	1366	1365		0.26035308837890%	0.26044845581050%	0.13027191162110%	0.13017654418950%	
6	0.39062500000000%	682	683				0.06504058837890%	0.06513595581050%			
8	0.09765625000000%	170	171				0.01621246337890%	0.01630783081050%			
10	0.02441406250000%	42	43				0.00400543212890%	0.00410079956050%			
12	0.00610351562500%	10	11				0.00095367431640%	0.00104904174800%			
14	0.00152587890630%	2	3				0.00019073486330%	0.00028610229490%			
16	0.00038146972660%	1	1				0.00009536743160%	0.00009536743160%			
18	0.00009536743160%	0	1				0.00000000000000%	0.00009536743160%			
Totale	8.33330154418950%	14559	3644	1366	1365	32768	1.38845443725590%	0.34751892089840%	0.13027191162110%	0.13017654418950%	3.12500000000000%
				Total numbers sliding --->		53702	Total probability of numbers sliding family D -->			5.12142181396484%	

Therefore, even the numbers belonging to the families C and D of a group of consecutive integers **between 1 and 2^n** for every large n (i.e. **12.5%)** that satisfy the two options of **Note 4.1** are **6.09%** of the group for which the remaining **6.41%** does not ***slip*** in the first 2 iterations or does not have a number less than which is isopath.
The following tables identify for the C and D families the values of w, the total number corresponding to the value of p and the value of w and the relative probability of the numbers that do not slip of the group of numbers between 1 and 2^20.

Family C - Case 6 with odd p and even k with N[0] = 2^p * (1 + 4*w) - 1					
Numbers that don't slip in the first 2 iterations					
p=	w=1+2*t	w=0+4*t	w=2+4*t	total numbers	Probability
3	16384		8192	24576	2.3437500%
5	4096	2048		6144	0.5859375%
7	1024	512	512	2048	0.1953125%
9	256	128	128	512	0.0488281%
11	64	32	32	128	0.0122070%
13	16	8	8	32	0.0030518%
15	4	2	2	8	0.0007629%
17	1	1		2	0.0001907%
19		1		1	0.0000954%
Total --->	21845	2732	8874	33451	3.1901360%

Family D - Case 6 with even p and odd k with N[0]=2^p*(3+4*w)-1							
Numbers that don't slip in the first 2 iterations							
p=	w=1+6*t	w=5+6*t	w=4+12*t	w=6+12*t	w=0+2*t	total numbers	Probability
2	10923	10923				21846	2.0833969%
4	2731	2730	1365	1364		8190	0.7810593%
6	342	341			2048	2731	0.2604485%
8	86	85			512	683	0.0651360%
10	22	21			128	171	0.0163078%
12	6	5			32	43	0.0041008%
14	2	1			8	11	0.0010490%
16	1				2	3	0.0002861%
18					1	1	0.0000954%
Total --->	14113	14106	1365	1364	2731	33679	3.2118797%

Furthermore, the values of the previous tables **remain the same** even for groups of numbers from 1 to 2^n for any value of n because:

1. families C and D are defined by modular and parametric conditions (on p – k – w , etc.) that are repeated regularly even in the interval [1; 2^n];
2. in the interval [1; 2^n] the percentages that are in the table (for each value of p for each block of "total numbers") are **densities**, not random absolute numbers;

3. if it extends to [1; 2^n], we are not changing the law that generates families C and D, but only **by repeating the same structure over a larger interval**.

So the **probabilities for each row** (the ones you have in the tables of C and D) and the **total sum** (the famous **6.41%** of the critical subset) are **stable for any value of n.**

5. The geometric decay model of the C and D families

In the Collatz custom sequence, the numbers belonging to the C and D families are the only candidates potentially able to increase the value of N[0] in the iterations following the second, but in the custom sequence there is a **93.59% probability of converging to numbers less than the initial number of each iteration.**

In this section we introduce a model of **geometric decay**, which quantitatively describes the progressive reduction of the density of the "dangerous" numbers of the C and D families in the iterations following the second. This model is a fundamental step to demonstrate that the custom sequence necessarily converges to 1.

We have seen that considering the range of numbers from 1 to 2^n for every positive integer n, the modular classification of families A, B, C, and D implies that the density of (**C**∪**D**) is 12.5% and that **6.09%** of the numbers of families C and D either slip within the first two iterations or have a smaller isopath number so that the **residue** of potentially dangerous numbers is p_0=6.41%, and this value represents the initial density of numbers that, after two iterations, have not yet been eliminated.

For each compound iteration subsequent to the second, it is observed that a number belonging to the C and D families has a 93.59% probability of falling below the initial value of the iteration or of having a smaller isopath number for which the complementary probability, i.e. the probability that the number **will not** slip in an iteration, is r=1- q = 0.0641. This probability represents the fraction of the numbers that remain "active" and potentially able to increase the value of N.

Let p_k the density of the numbers of the C and D families that do not slip after k iterations subsequent to the second. The process of reduction is described by the geometric progression:

$$p_k = p_0 \cdot r^k = 0{,}0641 \cdot (0{,}0641)^k.$$

The first values are:

$$p_0 = 0{,}0641,$$
$$p_1 = 0{,}0641^2 \approx 0{,}00411,$$
$$p_2 = 0{,}0641^3 \approx 0{,}00026,$$
$$p_3 = 0{,}0641^4 \approx 0{,}000017.$$

The density of dangerous numbers is therefore reduced extremely rapidly: already after three or four iterations following the second, the residual mass is negligible.

Theorem 5.1 (of geometric decay)) Let p_k = 0.0641 be the density of numbers from families C and D that do not slip within the first two iterations of the customized sequence. Let r = 0.0641 be the probability that such numbers do not slip in a subsequent iteration. Then the density of numbers that do not slip after k iterations is:

$$p_k = p_0 r^k.$$

In particular:

$$\lim_{k\to\infty} p_k = 0.$$

Proof. The sequence $\{p_k\}$ is a geometric progression with ratio r<1. Since 0<r<1, it immediately follows that:

$$\lim_{k\to\infty} p_0 r^k = 0.$$

Therefore, the density of numbers of non-slipping C and D households tends to zero.

6. Consequences for the convergence of personalized succession

The geometric decay model shows that the mass of numbers potentially capable of increasing the value of *N* decreases rapidly and tends to zero.

In addition, the structure of compound iterations ensures that the difference between even and odd steps always increases by 1 or 2 units, and this difference increases indefinitely along the sequence. Recall that in each iteration if the initial number of it is a case 3 or a case 5 (numbers of families A and B) at the end of the iteration the difference between even steps and odd steps is increased by 2 units, while if the initial number of it is a case 4 or a case 6 (numbers of families C and D) at the end of the iteration the difference between even and odd steps is increased by 1 unit. So after k iterations we have that the sum of $b_i - a_i$ with i from 1 to k turns out to be >=k so the growth is **linear**, and often **super-linear** if cases 3 or 5 appear in the iterations. The overall reduction factor after iterations is:k

$$\frac{2^{b_0+b_1+\cdots+b_k}}{3^{a_0+a_1+\cdots+a_k}}$$

and since $(b_0-a_0)+\cdots+(b_k-a_k)\geq k$, we have: $b_0+\cdots+b_k \geq a_0+\cdots+a_k+k$ and therefore

$$2^{b_0+\cdots+b_k} \geq 2^{a_0+\cdots+a_k+k}.$$

From which:

$$\frac{2^{b_0+\cdots+b_k}}{3^{a_0+\cdots+a_k}} \geq 2^k \left(\frac{2}{3}\right)^{a_0+\cdots+a_k}.$$

The term grows exponentially, while it decreases. So the overall reduction factor tends to $2^k \left(\frac{2}{3}\right)^{a_0+\cdots+a_k}$ **infinity**, so it follows that the reduction factor 2^b (where b indicates the even total steps) dominates the expansion factor 3^a (where a indicates the odd total steps) and therefore every number, after a finite number of iterations, must necessarily slip.

7. Conclusion

By combining this last result with Theorem 5.1 (of geometric decay), we get that geometric decay quickly reduces the mass of numbers that could go up, while the growth of *b-a* shows that every number, even the remaining 6.41%, has to go down after a finite number of iterations, so these two results together form the basis of the proof that the customized sequence inevitably converges to 1.

8. References

- Ardito Nicola — *Collatz's conjecture from an elementary point of view*. Collatz's conjecture from the elementary point of view
- Ardito Nicola — *Custom Collatz Sequence, Modular Reduction, Isopath Orbits, and Odd and Even Step Analysis*.
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